第三章 量子力学的数学表示：态与算符

1. 量子态 (完全可以用行列向量来理解)

a. $|\psi\rangle$ ket矢

prop. 线性叠加原理 $\longrightarrow$ 由所有可能的态 $\{|\psi_i\rangle\}$ 组成的复矢量空间为态空间 (Hilbert)

复数

运算规则：① $c|\psi\rangle = |\psi\rangle c$ ② $|\psi\rangle$ 与 $c|\psi\rangle$ 为同一量子态

b. $\langle\psi|$ bra矢 定义为 $|\psi\rangle$ 的共轭矢量 (dual) (不代表实际量子态，只是运算规则)

运算规则：① $(|\psi\rangle)^\dagger = \langle\psi|$ ② $(\langle\psi|)^\dagger = |\psi\rangle$ ③ $(c|\psi\rangle)^\dagger = \langle\psi|c^*$ ④ $(c\langle\psi|)^\dagger = c^*|\psi\rangle$

⑤ $(c_1|\psi_1\rangle + c_2|\psi_2\rangle)^\dagger = \langle\psi_1|c_1^* + \langle\psi_2|c_2^*$

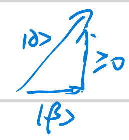
c. 内积 $|\alpha\rangle, |\beta\rangle \longrightarrow \langle\alpha|\beta\rangle$ 取复数 且要求
- ① $\langle\alpha|\beta\rangle = (\langle\beta|\alpha\rangle)^* \Rightarrow \langle\alpha,\alpha\rangle$ real
- ② $\langle\alpha|\alpha\rangle \geq 0$ 当仅当 $|\alpha\rangle$ 为零态
- ③ 线性

d. 直积 $|\alpha\rangle \otimes |\beta\rangle = |\alpha,\beta\rangle$ 将两独立自由度组装成新的态

两个量子数表示一个态

ex. (Schwartz) $|\langle\alpha|\beta\rangle| \leq \sqrt{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle}$

证：$(\langle\alpha| + \lambda^*\langle\beta|)\cdot(|\alpha\rangle + \lambda|\beta\rangle) \geq 0$ 令 $\lambda = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$ 则 $\lambda^* = -\frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}$



$$\Rightarrow \langle\alpha|\alpha\rangle - \frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}\langle\alpha|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}\langle\beta|\alpha\rangle + \frac{\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle}{(\langle\beta|\beta\rangle)^2}\langle\beta|\beta\rangle \geq 0$$

$$\Rightarrow \langle\alpha|\alpha\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}\langle\beta|\alpha\rangle \geq 0 \quad \checkmark$$

2. 算符

定义算符为作用于ket矢上的变换操作 $\hat{A}|\psi\rangle = |\varphi\rangle$ 线性映射

单位算符 $\hat{I}|\psi\rangle = |\psi\rangle$

prop. $\forall |\psi\rangle$，若 $\hat{A}|\psi\rangle = \hat{B}|\psi\rangle \Rightarrow \hat{A} = \hat{B}$ 推广：$\underset{\langle\psi|\cdot|\hat{A}|\psi\rangle}{\underset{||}{\langle\varphi|\hat{A}|\psi\rangle}} = \langle\psi|\hat{B}|\psi\rangle \Rightarrow \hat{A} = \hat{B}$

*算符的和 交换律 ✓

*算符的积 结合律 ✓ $\hat{A}\hat{B}|\psi\rangle = \hat{A}(\hat{B}|\psi\rangle)$ 一般无交换律！

*基本对易关系 $[\hat{x},\hat{p}] = i\hbar$ (可以当做假设) 3-dim：$[\hat{x}_\alpha, \hat{p}_\beta] = i\hbar\delta_{\alpha\beta}$

对易子：$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$
反对易子：$\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$

形式证明：(坐标空间、一维情况) $\hat{p} \to -i\hbar\frac{\partial}{\partial x}$ $\hat{x} \to x$

ex. $[\hat{A},\hat{B}] = -[\hat{B},\hat{A}]$ $[\hat{A},\hat{A}] = 0$ $[\hat{A},c] = 0$ $[\hat{A},\hat{B}+\hat{C}] = \hat{A}(\hat{B}+\hat{C}) - (\hat{B}+\hat{C})\hat{A} = [\hat{A},\hat{B}] + [\hat{A},\hat{C}]$

$[\hat{A},\hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{A}\hat{C} = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$

$[\hat{A}\hat{B},\hat{C}] = [\hat{A},\hat{C}]\hat{B} + \hat{A}[\hat{B},\hat{C}]$

$\boxed{\hat{A}\hat{B} = \hat{B}\hat{A} + \underbrace{[\hat{A},\hat{B}]}_{\text{Residue}}} \longrightarrow$ 可以用于对易的方式交换算符 (不断移动)

ex. 角动量算符的对易子

$\vec{L}=\hat{\vec{r}}\times\hat{\vec{p}}$ $\hat{\vec{r}}=\hat{x}\vec{e_x}+\hat{y}\vec{e_y}+\hat{z}\vec{e_z}$

$=\begin{vmatrix} e_x & e_y & e_z \\ \hat{x} & \hat{y} & \hat{z} \\ -i\hbar\frac{\partial}{\partial x} & -i\hbar\frac{\partial}{\partial y} & -i\hbar\frac{\partial}{\partial z}\end{vmatrix}=\varepsilon_{\alpha\beta\gamma}\hat{x}_\alpha\hat{p}_\beta\vec{e_\gamma}$

$\Rightarrow [\hat{L}_x,\hat{x}]=0$ $[\hat{L}_x,\hat{y}]=[\hat{y}\hat{p}_z-\hat{z}\hat{p}_y,\hat{y}]=-\hat{z}[\hat{p}_y,\hat{y}]=i\hbar\hat{z}$ 对易

$\Rightarrow [\hat{L}_\alpha,\hat{x}_\beta]=[\varepsilon_{\gamma\lambda\alpha}\hat{x}_\gamma\hat{p}_\lambda,\hat{x}_\beta]=\varepsilon_{\gamma\lambda\alpha}\hat{x}_\gamma(-i\hbar)\delta_{\lambda\beta}=i\hbar\varepsilon_{\alpha\beta\gamma}\hat{x}_\gamma$

$[\hat{L}_\alpha,\hat{p}_\beta]=i\hbar\varepsilon_{\alpha\beta\gamma}\hat{p}_\gamma$ $[\hat{L}_\alpha,\hat{L}_\beta]=i\hbar\varepsilon_{\alpha\beta\gamma}\hat{L}_\gamma$

* 算符的逆 若 $\hat{A}|\psi\rangle=|\varphi\rangle$ 则 $\hat{A}^{-1}|\varphi\rangle=|\psi\rangle$

$\Rightarrow \hat{A}\hat{A}^{-1}=\hat{I}$ $(\hat{A}\hat{B})^{-1}=\hat{B}^{-1}\hat{A}^{-1}$

* 算符的幂

* 算符的函数 $f(\hat{A})=\sum_{n=0}^{+\infty}\frac{f^{(n)}(0)}{n!}\hat{A}^n$

* 算符作用在bra矢上 $\langle\psi|\hat{A}=\langle\varphi|$ $(\langle\psi|\hat{A})^+=\hat{A}^+|\psi\rangle$

① $\langle\varphi|\hat{A}^+|\psi\rangle=\langle\varphi|(\langle\psi|\hat{A})^+\overset{\text{dot prop.}}{=\!=}((\langle\psi|\hat{A})|\varphi\rangle)^*=\langle\psi|\hat{A}|\varphi\rangle^*$

② $(\hat{A}^+)^+=\hat{A}$

③ $\langle\varphi|A^+=(\hat{A}|\varphi\rangle)^*$

④ $(\hat{A}\hat{B})^+|\varphi\rangle=(\langle\varphi|\hat{A}\hat{B})^+=\hat{B}^+(\langle\varphi|\hat{A})^+=\hat{B}^+\hat{A}^+|\varphi\rangle$ $\Rightarrow (\hat{A}\hat{B})^+=\hat{B}^+\hat{A}^+$

* 外积. $|\alpha\rangle\langle\beta|$ 作为算符 $(|\alpha\rangle\langle\beta|)|\varphi\rangle=|\alpha\rangle\langle\beta|\varphi\rangle$

prop. $(|\alpha\rangle\langle\beta|)^+=|\beta\rangle\langle\alpha|$

例: 广义结合律

$\langle\psi_1|\hat{A}|\varphi_1\rangle\langle\psi_2|\varphi_2\rangle=\langle\psi_1|\hat{A}(|\varphi_1\rangle\langle\psi_2|)\varphi_2$

例: 密度矩阵算符 → 可以代表系统状态 (纯.混都可)

$\hat{\rho}=|\psi\rangle\langle\psi|$ (only 纯态) $(|\psi\rangle\langle\psi|)|\varphi\rangle=\langle\psi|\varphi\rangle|\psi\rangle$ 投影算符

在叠加态时 $\hat{\rho}=(\alpha|1\rangle+\beta|2\rangle)(\langle1|\alpha^*+\langle2|\beta^*)$

$=|\alpha|^2|1\rangle\langle1|+|\beta|^2|2\rangle\langle2|+\alpha\beta^*|1\rangle\langle2|+\alpha^*\beta|2\rangle\langle1|$

相干项 → 纠缠态

若为0则为混态 (经典概率)

3. 厄米算符 —— 所有可观测量对应算符

def: $\hat{A}^+=\hat{A}$

prop. ① $\langle\varphi|\hat{A}|\psi\rangle=(\langle\psi|\hat{A}|\varphi\rangle)^*$ ② $\hat{A}+\hat{B}$ 也是Hermitian ($\hat{A}\hat{B}$不一定是Hermitian)

③ Hermitian 本征值为实数, 对应不同本征值的非简并本征态相互正交

算符的本征态: $\hat{A}|\psi_n\rangle=A_n|\psi_n\rangle\Rightarrow\langle\psi_n|\hat{A}^+=\langle\psi_n|A_n^*$

$$\begin{cases} \langle\psi_m|\hat{A}|\psi_n\rangle = A_n\langle\psi_m|\psi_n\rangle \\ \langle\psi_m|\hat{A}|\psi_n\rangle = \langle\psi_m|A_n^*|\psi_n\rangle \end{cases} \xrightarrow{\psi_m,\psi_n\text{均不为零态}} (A_n-A_n^*)\langle\psi_m|\psi_n\rangle=0 \Rightarrow \begin{cases} m=n & A_n=A_n^* \\ m\neq n & \langle\psi_m|\psi_n\rangle=0 \end{cases}$$

④ 归一化后 Hermitian 本征态集合构成态空间的一组正交完备基 (ps: 不是所有算符都可以)

$\langle\psi_m|\psi_n\rangle=\delta_{mn}$

离散：$|\psi\rangle=\sum_n C_n|\psi_n\rangle = \sum_n\langle\psi_n|\psi\rangle|\psi_n\rangle = \sum_n|\psi_n\rangle\langle\psi_n|\psi\rangle = \left(\sum_n|\psi_n\rangle\langle\psi_n|\right)|\psi\rangle$

$\Leftrightarrow$ 完备性：$\sum_n|\psi_n\rangle\langle\psi_n| = \hat{I}$

连续：$\{|r\rangle\}$　　正交归一：$\langle r|r'\rangle=\delta^3(r-r')$

完备性：$|\psi\rangle=\int d^3r\,\psi(r)|r\rangle = \int d^3r\,\langle r|\psi\rangle|r\rangle = \left(\int d^3r\,|r\rangle\langle r|\right)|\psi\rangle$

$\Leftrightarrow \int d^3r\,|r\rangle\langle r| = \hat{I}$

ex. $\langle\psi|\psi\rangle=1 \xrightarrow{\text{设}|\psi\rangle=\sum_n C_n|\psi_n\rangle} \sum_{nm} C_m^*C_n\langle\psi_m|\psi_n\rangle=1 \Rightarrow \sum_n|C_n|^2=1$ (统计解释要求)

or $1=\langle\psi|\psi\rangle = \langle\psi|\sum_n|\psi_n\rangle\langle\psi_n|\psi\rangle = \sum_n C_n^*C_n = \sum_n|C_n|^2$

ex. $\langle\psi|\psi\rangle=1=\langle\psi|\left(\int d^3r\,|r\rangle\langle r|\right)|\psi\rangle = \int d^3r\,\langle\psi|r\rangle\langle r|\psi\rangle = \int d^3r\,|\psi(r)|^2$

综上，故$|\psi\rangle$可归一化等价于任何表象下态都在 Hilbert 空间里

ex. $\bar{A}=\sum_n|C_n|^2A_n = \sum_n A_n\langle\psi|\psi_n\rangle\langle\psi_n|\psi\rangle = \sum_n\langle\psi|A_n|\psi_n\rangle\langle\psi_n|\psi\rangle = \sum_n\langle\psi|\hat{A}|\psi_n\rangle\langle\psi_n|\psi\rangle = \langle\psi|\hat{A}|\psi\rangle$

⑤ 任意量子态下，厄米算符期望值为实数

$\bar{A}=\langle\psi|\hat{A}|\psi\rangle = \left(\langle\psi|\hat{A}^+|\psi\rangle\right)^* = \bar{A}^*$

⑥ 任意量子态下，期望值为实数的算符是厄米算符

证明：令 $|\psi\rangle=|\psi_1\rangle+c|\psi_2\rangle$

$$\underset{R}{\bar{A}}=\langle\psi|\hat{A}|\psi\rangle = \underset{R}{\langle\psi_1|\hat{A}|\psi_1\rangle} + \underset{R}{|c|^2\langle\psi_2|\hat{A}|\psi_2\rangle} + c\langle\psi_1|\hat{A}|\psi_2\rangle + c^*\langle\psi_2|\hat{A}|\psi_1\rangle$$

令 $c=1$，有 $\langle\psi_1|\hat{A}|\psi_2\rangle+\langle\psi_2|\hat{A}|\psi_1\rangle = ①$

令 $c=i$，有 $i\langle\psi_1|\hat{A}|\psi_2\rangle - i\langle\psi_2|\hat{A}|\psi_1\rangle = ②$

$\Rightarrow \langle\psi_1|\hat{A}|\psi_2\rangle = \dfrac{①+i②}{2}$

$\langle\psi_2|\hat{A}|\psi_1\rangle = \dfrac{①-i②}{2} = \langle\psi_1|\hat{A}|\psi_2\rangle^* = \langle\psi_2|\hat{A}^+|\psi_1\rangle$

由于 ψ_2,ψ_1 任意 $\Rightarrow \hat{A}=\hat{A}^+$

推论：可观测量的单次测值和期望值为实数，所以可观测量对应算符必为厄米算符

ex. $\hat{P}_x$的本征态求法（坐标表示）

$$\hat{P}_x|P_x\rangle = P_x|P_x\rangle \Rightarrow -i\hbar\frac{\partial}{\partial x}\psi_{p_x}(x) = P_x\psi_{p_x}(x) \Rightarrow \psi_{p_x}(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{iP_x x}{\hbar}}$$ 平面波

ex. 自由粒子的能量本征态

$$\hat{H}|\psi_E\rangle = E|\psi_E\rangle \Rightarrow -\frac{\hbar^2\nabla^2}{2m}\psi_E = E\psi_E \Rightarrow \psi_E = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}e^{i\vec{p}\cdot\vec{r}/\hbar}$$ 平面波

$$\sqrt{\frac{n}{m}},$$

4. 简谐振子的代数解法

① 一维振子的能量本征问题

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

ladder operator 升降算符

定义 $\begin{cases} \hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} + \frac{i}{\sqrt{m\omega\hbar}}\hat{p}\right) \\ \hat{a}^+ = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} - \frac{i}{\sqrt{m\omega\hbar}}\hat{p}\right) \end{cases}$ （$\hat{a}$不是厄米的）

$$\Rightarrow \begin{cases} \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^+) \\ \hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^+) \end{cases}$$

由$[\hat{x},\hat{p}] = i\hbar \Rightarrow [\hat{a},\hat{a}^+] = 1$

$$\Rightarrow \begin{cases} \hat{x}^2 = \frac{\hbar}{2m\omega}\left(\hat{a}^2 + (\hat{a}^+)^2 + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right) \\ \hat{p}^2 = -\frac{m\omega\hbar}{2}\left(\hat{a}^2 + (\hat{a}^+)^2 - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a}\right) \end{cases}$$

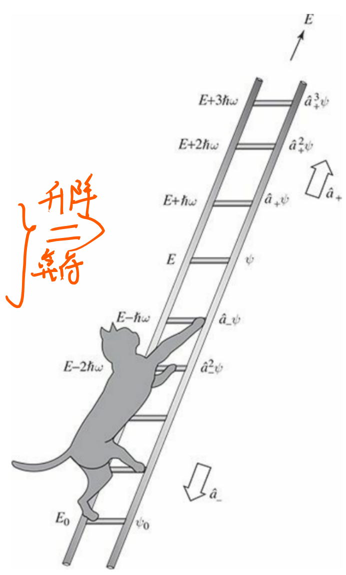
$$\Rightarrow \hat{H} = \left(\hat{a}^+\hat{a} + \frac{1}{2}\right)\hbar\omega$$

粒子数算符

若再定义 $\hat{N} = \hat{a}^+\hat{a}$ 则 $\hat{H} = (\hat{N} + \frac{1}{2})\hbar\omega$ $\Rightarrow [\hat{N},\hat{a}] = -\hat{a}$ $[\hat{N},\hat{a}^+] = \hat{a}^+$

$\hat{N}$是厄米的，设 $\hat{N}|n\rangle = n|n\rangle$ 则 $\hat{H}|n\rangle = \underbrace{(n+\frac{1}{2})\hbar\omega}_{E_n}|n\rangle$

Fock态



$\hat{N}(\hat{a}^+|n\rangle) = (\hat{a}^+\hat{N} + \hat{a}^+)|n\rangle = (\hat{a}^+n + \hat{a}^+)|n\rangle = (n+1)(\hat{a}^+|n\rangle) \xRightarrow{\text{假设}|n+1\rangle\text{非简并}} \boxed{\hat{a}^+|n\rangle = C|n+1\rangle}$

$\hat{N}(\hat{a}|n\rangle) = (\hat{a}\hat{N} - \hat{a})|n\rangle = (n-1)(\hat{a}|n\rangle) \xRightarrow{\cdots\cdots} \boxed{\hat{a}|n\rangle = D|n-1\rangle}$

升降算符

下面求C, D：

$$\langle n|\hat{a}^+\hat{a}|n\rangle = n \Rightarrow |D|^2 = n$$
$$\langle n|\hat{a}^+\hat{a}|n\rangle = (\hat{a}|n\rangle)^+(\hat{a}|n\rangle)$$

规范对称性

设C, D为实数（比例没有可观测影响）

$$\langle n|\hat{a}\hat{a}^+|n\rangle = n+1 \Rightarrow |C|^2 = n+1$$
$$\langle n|\hat{a}\hat{a}^+|n\rangle = (\hat{a}^+|n\rangle)^+(\hat{a}^+|n\rangle)$$

$$\Rightarrow \begin{cases} \hat{a}|n\rangle = \sqrt{n}|n-1\rangle & \text{湮灭算符} \\ \hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle & \text{产生算符} \end{cases}$$

* 连续作用 $\hat{a}^-$ 于 $|n\rangle$ 上

$\hat{a}^-|n_0+1\rangle = \sqrt{n_0+1}\,|n_0\rangle$　　要求　$n = \langle n|\hat{a}^+\hat{a}^-|n\rangle \geq 0$（模必须不能小于0）

故若 $n \in Z^+$，则 $n_0=0$；若 $n \notin n^+$，则 $0<n_0<1$，but $\hat{a}^-|n_0\rangle$ 无意义，与上式矛盾

$\Rightarrow$　$n_0 = 0$　　n为非负整数

* $E_n = (n+\frac{1}{2})\hbar\omega$　　$n=0,1,2,\cdots$　　离散能谱（一份一份 $\hbar\omega$）

n为粒子数

$n=0$ 时 基态　$E_0 = \frac{1}{2}\hbar\omega$（真空能）　量子涨落　不确定关系　$|0\rangle$

* $|n\rangle = \dfrac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$

* $\begin{cases} \hat{x}|n\rangle = \sqrt{\dfrac{\hbar}{2m\omega}}\left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right) \\ \hat{p}|n\rangle = -i\sqrt{\dfrac{m\omega\hbar}{2}}\left(\sqrt{n}\,|n-1\rangle - \sqrt{n+1}\,|n+1\rangle\right) \end{cases}$

$\Rightarrow$　$\bar{x} = \langle n|\hat{x}|n\rangle = 0$　　　$\bar{p} = 0$

$\Rightarrow$　$\Delta x \Delta p = (n+\frac{1}{2})\hbar$　　基态具有最小不确定关系

② Fock态的波函数

从 $\langle x|0\rangle$ 入手

$\langle x|\hat{a}|0\rangle = 0$　　$\Rightarrow$　$\sqrt{\dfrac{m\omega}{2\hbar}}\langle x|(\hat{x} + \dfrac{i}{m\omega}\hat{p})|0\rangle = 0$

$\langle x|\hat{x}|0\rangle = x\langle x|0\rangle$

$\langle x|\hat{p}|\hat{0}\rangle = \int dx' \langle x|\hat{p}|x'\rangle\langle x'|0\rangle$
$= \int dx' \left(-i\hbar\frac{\partial}{\partial x}\delta(x-x')\right)\langle x'|0\rangle = -i\hbar\frac{\partial}{\partial x}\psi_0(x)$

in position space：$\hat{p} \to -i\hbar\frac{\partial}{\partial x}$　$\hat{x} \to x$　$\Longrightarrow$　$\left(x + \dfrac{\hbar}{m\omega}\dfrac{d}{dx}\right)\psi_0(x) = 0$

$\Rightarrow$　$\psi_0(x) = \dfrac{1}{\pi^{\frac{1}{4}}\sqrt{x_0}}\, e^{-\frac{x^2}{2x_0^2}}$　　其中 特征长度 $x_0 = \sqrt{\dfrac{\hbar}{m\omega}}$

Gauss wave packet

$\Rightarrow \psi_n(x) = \dfrac{1}{\pi^{\frac{1}{4}}\sqrt{2^n n!}}\left(\dfrac{1}{x_0^{n+\frac{1}{2}}}\right)\left(x - x_0^2\dfrac{\partial}{\partial x}\right)^n e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}$

厄米多项式

③ 相干态

$\hat{a}$ 的本征态　　$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$　　　α一般为复数，相位很重要，一般 α 连续取值

i) 相干态与Fock态关系

$|\alpha\rangle = \sum\limits_n |n\rangle\langle n|\alpha\rangle = \sum\limits_n |n\rangle \dfrac{1}{\sqrt{n!}}\langle 0|a^n|\alpha\rangle = \sum\limits_n \dfrac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle\,|n\rangle$

由 $1 = \langle\alpha|\alpha\rangle = \sum\limits_n \langle\alpha|n\rangle\langle n|\alpha\rangle = \sum\limits_n \dfrac{1}{n!}|\alpha|^{2n}|\langle 0|\alpha\rangle|^2 = e^{|\alpha|^2}|\langle 0|\alpha\rangle|^2 \xRightarrow{\text{取}\langle 0|\alpha\rangle \in R} \langle 0|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}$

$$\Rightarrow \quad |\alpha\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{1}{2}|\alpha|^2} |n\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \frac{(a^+)^n}{\sqrt{n!}} |0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^+} |0\rangle$$

ii) 相干态的归一性(验证)

$$\langle\alpha|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{nm} \frac{\alpha^n(\alpha^*)^m}{\sqrt{n!m!}} \langle m|n\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} = 1$$

iii) 正交完备性

$$\langle\alpha|\beta\rangle = \sum_{mn} \frac{(\alpha^*)^m \beta^n}{\sqrt{m!n!}} \langle m|n\rangle e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2} = e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta}$$ 非正交

$$\int d^2\alpha \, |\alpha\rangle\langle\alpha| = \int d^2\alpha \, e^{-|\alpha|^2} \sum_{mn} \frac{\alpha^n(\alpha^*)^m}{\sqrt{n!m!}} |n\rangle\langle m|$$ 令 $\alpha = \rho e^{i\varphi}$

$$= \int \rho \, d\rho \, d\varphi \, e^{-\rho^2} \sum_{mn} \frac{\rho^{m+n}}{\sqrt{n!m!}} e^{i\varphi(n-m)} |n\rangle\langle m|$$

$$= 2\pi \int \rho \, d\rho \, e^{-\rho^2} \sum_n \frac{\rho^{2n}}{n!} |n\rangle\langle n|$$

$$= \pi \sum_n |n\rangle\langle n| = \pi \hat{I}$$ 超完备性

iv) 不确定关系 ($|\alpha\rangle$)

定义广义坐标动量 (无量纲) $\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^+)$ $\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^+)$

对易交换

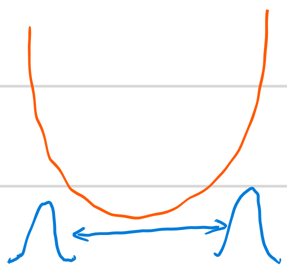
$\hat{X}_1^2 = \frac{1}{4}(\hat{a}^2 + \hat{a}^{+2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a})$ $\hat{X}_2^2 = \cdots$

$$\langle\alpha|\hat{X}_1|\alpha\rangle = \frac{1}{2}(\alpha + \alpha^*) = \mathrm{Re}\,\alpha \qquad \langle\alpha|\hat{X}_1^2|\alpha\rangle = \frac{1}{4}(\alpha^{*2} + \alpha^2 + 2\alpha^*\alpha + 1) = (\mathrm{Re}\,\alpha)^2 + \frac{1}{4}$$

$$\langle\alpha|\hat{X}_2|\alpha\rangle = \frac{1}{2i}(\alpha - \alpha^*) = \mathrm{Im}\,\alpha \qquad \langle\alpha|\hat{X}_2^2|\alpha\rangle = -\frac{1}{4}(\alpha^2 + (\alpha^*)^2 - 2|\alpha|^2 - 1) = (\mathrm{Im}\,\alpha)^2 + \frac{1}{4}$$

$\Rightarrow \quad \Delta X_1 = \frac{1}{2} \qquad \Delta X_2 = \frac{1}{2} \qquad \Delta X_1 \Delta X_2 = \frac{1}{4}$ (最小不确定度) ← 与α取值无关

$|\alpha\rangle$是最像经典态的量子态 (波函数含时演化)



波包振荡,像一个振子

v) 粒子数

$$\langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|\hat{a}^+\hat{a}|\alpha\rangle = |\alpha|^2 \qquad P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$ poisson

vi) 三维谐振子

$$\hat{H} = \sum_{i=x,y,z} \left(\frac{\hat{p}_i^2}{2m} + \frac{1}{2} m\omega^2 \hat{r}_i^2 \right)$$

定义 $\hat{a}_x, \hat{a}_y, \hat{a}_z$ $\Rightarrow$ $\hat{H} = (\hat{N}_x + \hat{N}_y + \hat{N}_z) + \frac{3}{2}\hbar\omega$ 本征态 $|n_x, n_y, n_z\rangle$ $E_{n_x,n_y,n_z} = (n_x + n_y + n_z + \frac{3}{2})\hbar\omega$